CLASSIFICATION OF LINEAR DIFFERENTIAL OPERATORS WITH AN INVARIANT SUBSPACE OF MONOMIALS

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ABSTRACT. A complete classification of linear differential operators possessing finite-dimensional invariant subspace with a basis of monomials is presented.

1. Introduction

One of the old-standing problems in the theory of special functions is the classification of all linear differential operators that admit an infinite sequence of orthogonal eigenvectors in the form of polynomials. See [2] for an overview. In 1929 S. Bochner [1] had solved this problem for second order differential operators on the complex (real) line. Therefore we name this problem the "Bochner problem".

The main purpose of the present paper is to study a more general problem, which we name (following [4]) the "generalized Bochner problem". We ask for a classification of all linear differential operators, which possess a **finite-dimensional** invariant subspace of polynomials. It turns out this problem is rather sophisticated. However, things get much more tangible when, instead of invariant subspace in polynomials, we require that the (finite-dimensional) invariant subspace has a basis of monomials. This problem is solved completely (Section 3). In [4] a particular case of this problem was solved, namely if the invariant subspace is the linear space of polynomials of degree not higher than some fixed integer. The classification of the linear operators possessing such an invariant subspace was given through the universal enveloping algebra of the algebra sl_2 taken in a special representation by first-order differential operators.

The results of Section 3 have an impact to the general problem. This is presented in Section 4. In Section 5, we give the explicit expressions for the second-order differential operators T_2, possessing a finite-dimensional invariant subspace in monomials. These operators are of great interest for finding explicit solutions to the Schrödinger equation

$$(-\frac{d^2}{dx^2} + V(x))\Psi(x) = \lambda\Psi(x) \tag{1.1}$$

since the eigenvalue problem for the operator $T_2, T_2\varphi = \lambda\varphi$ can be reduced to the Schrödinger equation by a change of the variable, $x' = x'(x)$, and introducing a new

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(gauge-transformed) function, $\Psi = \varphi \exp(-a(x))$. Hence the operators described in Section 5 lead to a special class of quasi-exactly-solvable Schrödinger equations complementary those described already [4].

Finally we make some concluding remarks in Section 6 concerning the case where the powers of x are not natural numbers.

Though in this paper we work over the complex numbers, the main results hold for real numbers as well.

2. Generalities

Here we state some general results concerning differential operators which leave a finite-dimensional subspace of $\mathbb{C}[x]$ invariant. The results of this section can easily be extended to more variables. Let $\mathfrak{D}$ denote the algebra of linear (finite-order) differential operators on $\mathbb{C}[x]$ with polynomial coefficients. We denote the symbol $\dfrac{d}{dx}$ by ∂.

Let V be a finite-dimensional subspace in $\mathbb{C}[x]$ of dimension n, and let $T : \mathbb{C}[x] \to \mathbb{C}[x]$ be a linear operator. It is easy to show that any linear operator T can be written as an infinite order linear differential operator:

$$T = \sum_{i=0}^{\infty} P_i \partial^i, \qquad P_i \in \mathbb{C}[x].$$

This can be proved easily by an inductive construction:

$$P_0 = T(1), \quad P_1 = T(x) - xP_0, \ldots .$$

It follows immediately that the action of T on V can be represented by a finite-order differential operator, since $V \subset I\!\!P_k$ (where $I\!\!P_k$ denotes the space of polynomials of degree not higher than k, and k sufficiently large), and hence for T_k:

$$T_k = \sum_{i=0}^{k} P_i \partial^i,$$

we have that $T_k(v) = T(v)$ for all $v \in V$.

So the following proposition is immediate:

Proposition 2.1. *Let $V \subset \mathbb{C}[x]$ be a subspace of dimension n, and let $\mathfrak{D}_V$ denote the algebra of differential operators that leave V invariant (i.e. $T_k \in \mathfrak{D}_V \Rightarrow T_k(V) \subset V$). Then $\mathfrak{D}_V$ is isomorphic to the semi-direct product of* $\mathrm{End}(V)$ *and I, where* $\mathrm{End}(V)$ *is the algebra of linear operators on V, and the ideal I is the algebra of differential operators which annihilate V.* □

We can paraphrase this proposition in the following way. As representation on V, $\mathfrak{D}_V$ is the full $n \times n$-matrix algebra, and the kernel of this representation is an infinite dimensional ideal in $\mathfrak{D}_V$.

Another consequence of this observation is, that for any element $v \in \mathbb{C}[x]$ and $\lambda \in \mathbf{C}$, we can find a differential operator T such that $Tv = \lambda v$.

In this paper, we mainly discuss the case that V is graded. Now $\mathbb{C}[x]$ is a graded algebra by putting $\deg(x^k) = k$. Let us take V a graded subspace. This means exactly that V has a basis of monomials. Hence we assume that $V = \langle x^{i_1}, x^{i_2}, \ldots , x^{i_n} \rangle$, which we abbreviate to $V = \langle x^I \rangle$, $I = \{i_1, i_2, \ldots , i_n\}$.

The fact that $\mathbb{C}[x]$ is graded has as a consequence that $\mathrm{End}(\mathbb{C}[x])$ is also graded, putting $\deg(T) = m$, if $T(x^i) \in \langle x^{i+m} \rangle$ for all i. Moreover for graded V it follows that $\mathfrak{D}_V$ is also graded. So, in order to describe the structure of $\mathfrak{D}_V$, it is sufficient to describe the homogeneous components of $\mathfrak{D}_V$. This is the concern of the next section.

3. Algebras leaving a space of monomials invariant

As before, we are interested in finite-order linear differential operators T, such that $T(V) \subset V$. We assume T to be graded of degree m and order k, which means that

$$T = \sum_{i=0}^{k} c_i x^{i+m} \partial^i.$$

Here $c_k \neq 0$ and $m \in \mathbb{Z}$. Moreover $c_i = 0$ for $i + m < 0$. In particular, we see that if the degree of T is negative, say $\deg(T) = -m, m > 0$, then the order of T is at least m.

The following lemma plays a crucial role in our classification.

Lemma 3.1. *Let T be a differential operator of degree m and order k.*

(1) *Suppose $m \geq 0$. Then there exist numbers $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ such that*

$$T = c_k x^m (x\partial - \alpha_1)(x\partial - \alpha_2) \dots (x\partial - \alpha_k).$$

(2) *Suppose $m < 0$, hence $k \geq -m$. Then there exist numbers $\alpha_1, \dots, \alpha_{k+m} \in \mathbb{C}$ such that*

$$T = c_k \partial^{-m} (x\partial - \alpha_1)(x\partial - \alpha_2) \dots (x\partial - \alpha_{k+m}).$$

Proof. If $m \geq 0$ it is clear that any differential operator of degree m is of the form $T = x^m \sum_{i=0}^{k} c_i x^i \partial^i$. Similarly, if $m < 0$, T can be put in the form $T = \partial^{-m} \sum_{i=0}^{k} \tilde{c}_i x^i \partial^i$ with $\tilde{c}_k = c_k$. So it suffices to prove the lemma for $m = 0$. But then we have $T(x^\alpha) = P_k(\alpha) x^\alpha$, where P_k is a k-th order polynomial. So $P_k(\alpha) = c_k(\alpha - \alpha_1)(\alpha - \alpha_2) \cdots (\alpha - \alpha_k)$ for some $\alpha_i \in \mathbb{C}$. Now

$$((x\partial - \alpha_1) \cdots (x\partial - \alpha_k))(x^\alpha) = (\alpha - \alpha_1)(\alpha - \alpha_2) \cdots (\alpha - \alpha_k),$$

From this it follows that $T = c_k (x\partial - \alpha_1)(x\partial - \alpha_2) \cdots (x\partial - \alpha_k)$, since the representation of $\mathfrak{D}$ on $\mathbb{C}[x]$ is faithful. □

It is easy to see that this factorization is unique up to the ordering of the factors $x\partial - \alpha_i$. However, a certain ordering of the factors has no meaning, since these factors commute.

Above representation of T given by lemma 3.1 is very convenient, when we study the operators T which leave $V = \langle x^{i_1}, \dots, x^{i_n} \rangle$ invariant, i.e. $T \in \mathfrak{D}_V$. We introduce the following notation. For $I = \{i_1, i_2, \dots, i_n\}$ and $m \in \mathbb{Z}$, we put

$$I^{(m)} = \{ i \in I \mid i + m \geq 0 \text{ and } i + m \notin I \}.$$

If $\deg(T) = m$ and $T \in \mathfrak{D}_V$, then it is clear that with necessity $T(x^i) = 0$ for $i \in I^{(m)}$, since $T(x^i) = c \cdot x^{i+m}$, but $x^{i+m} \notin V$, so $c = 0$. This leads to the following

Theorem 3.2. *Let* $V = \langle x^{i_1}, \dots, x^{i_n} \rangle$ *and* T *a finite-order differential operator such that* $\deg(T) = m$. *Suppose* $I^{(m)} = \{\alpha_1, \dots, \alpha_k\}$. *Then* $T \in \mathfrak{D}_V$, *if and only if*

$$T = \widetilde{T} \cdot (x\partial - \alpha_1)(x\partial - \alpha_2) \cdots (x\partial - \alpha_k),$$

where $\widetilde{T}$ *is some differential operator of degree* m.

Proof. The if-part is trivial. So assume that $T \in \mathfrak{D}_V$, and suppose it has order s. According to lemma 3.1, for $m \geq 0$, T can be represented in the form

$$T = c \cdot x^m (x\partial - \beta_1)(x\partial - \beta_2) \cdots (x\partial - \beta_s) \qquad (c \in \mathbb{C}, c \neq 0).$$

We need that $T(x^i) = 0$ for $i \in I^{(m)}$. On the other hand, we have

$$T(x^i) = c(i - \beta_1)(i - \beta_2) \cdots (i - \beta_s) x^{i+m}.$$

Hence it follows that $\{\alpha_1, \dots, \alpha_k\} \subset \{\beta_1, \dots, \beta_s\}$. After rearranging the β's we find

$$T = c \cdot x^m (x\partial - \beta_1)(x\partial - \beta_2) \cdots (x\partial - \beta_{s-k})(x\partial - \alpha_1) \cdots (x\partial - \alpha_k).$$

So for $m \geq 0$ the proposition is proved; $\widetilde{T} = c \cdot x^m (x\partial - \beta_1)(x\partial - \beta_2) \cdots (x\partial - \beta_{s-k})$. For $m < 0$ the proof is similar. □

REMARK. From the previous proposition it follows that the order of $T \in \mathfrak{D}_V$ with $\deg(T) = m$ is at least k, where k is the number of elements in $I^{(m)}$. In fact, up to a scalar coefficient, the element of order k is unique:

$$T = x^m (x\partial - \alpha_1) \cdots (x\partial - \alpha_k),$$

where $\{\alpha_1, \dots, \alpha_k\} = I^{(m)}$.

If $m = 0$, $I^{(m)} = \emptyset$, and we have $T = 1$. So all differential operators of degree 0 are in $\mathfrak{D}_V$; these elements form a commutative subalgebra of $\mathfrak{D}_V$ generated by $x\partial$.

4. The case of polynomial subspaces

In section 3, we performed the classification of differential operators with an invariant subspace V that has a basis of monomials. If V has no basis of monomials, $\mathfrak{D}_V$ is not be graded, but only filtered. This causes a major difficulty. However, considering the corresponding grading, we still can deduce some properties of $\mathfrak{D}_V$ in this case.

So, let V have a basis of the form

$$x^{i_1} + c_{11} x^{i_1 - 1} + \dots, \quad x^{i_2} + c_{21} x^{i_2 - 1} + \dots, \quad \dots \quad , \quad x^{i_n} + c_{n1} x^{i_n - 1} + \dots$$

We can assume that all i_j are different. The graded space $V(g)$ associated to V is $\langle x^{i_1}, \dots, x^{i_n} \rangle$.

Let $T \in \mathfrak{D}$ of order k be of the form

$$T = \sum_{i=-k}^{m} T^{(i)}$$

with $T^{(i)}$ of degree i, and $T^{(m)} \neq 0$. (An operator of order k has degree $-k$ or higher, the term of degree $-k$ being a multiple of ∂^k). We call $T^{(m)}$ the associated graded operator. Now it is easy to prove

Theorem 4.1. *Let V, $V(g)$, T and $T^{(m)}$ be as above.*
If $T \in \mathfrak{D}_V$, then $T^{(m)} \in \mathfrak{D}_{V(g)}$.

Proof. Suppose $T \in \mathfrak{D}_V$, and consider $T(x^{i_j} + c_{j1}x^{i_j-1} + \dots)$. If $m = 0$ there is nothing to prove, so consider $m \neq 0$. Let $I = \{i_1, i_2, \dots, i_n\}$. If $i_j \in I^{(m)}$ then $T^{(m)}(x^{i_j})$ should be 0, since no term of T can cancel this term. Hence we find exactly that $T^{(m)}(x^{i_j}) = 0$ for all $i_j \in I^{(m)}$, i.e. $T^{(m)} \in \mathfrak{D}_{V(g)}$. □

From this we derive an easy corollary:
If $T_k \in \mathfrak{D}$ of order k possesses an infinite number of (linearly independent) eigenvectors, then $T = \sum_{i=-k}^{0} T^{(i)}$, degree of $T^{(i)} = i$ and $T^{(0)} \neq 0$.

A similar reasoning can be performed for the part of T with minimal degree, but this seems to give not much information.

5. Classification of second order differential operators

5.1. Generic situation. Now we proceed to 2-nd order differential operators T_2, which admit a finite-dimensional invariant space of polynomials with a basis of monomials. We are interested in this problem in connection with finding explicit solutions to the Schrödinger equation (1.1). As mentioned in the Introduction, this involves some transformations see [3]. These transformations are of 2 types: the first is a change of basis, and the second is called "gauge" transformation, which amounts to changing T to gTg^{-1}, where g is a non-zero function. To delete some ambiguity in our spaces of monomials, we impose 2 conditions on $V = \{x^{i_1}, x^{i_2}, \dots, x^{i_n}\}$:

(1) We assume that $1 \in V$ (it removes an ambiguity resulting from gauge transformations).
(2) We assume that $\gcd(i_1, i_2, \dots, i_n) = 1$, i.e. that the powers have no common factor (it removes an ambiguity resulting from changes of variable).

For the classification of differential operators in $\mathfrak{D}_V$ these assumptions do not make much difference. If $T_2 = \sum T^{(i)}$, where the degree of $T^{(i)}$ is i, these two assumptions effect only the terms $T^{(-1)}$ and $T^{(-2)}$.

So let us start the classification. Suppose $T_2 \in \mathfrak{D}_V$, $T_2 = \sum_m T^{(m)}$ where $T^{(m)}$ has degree m and order less or equal to 2. If $T^{(m)}$ is non-zero, then according to proposition 3.2, $I^{(m)}$ contains 0,1 or 2 elements. We distinguish these 4 cases:

(1) For all $m > 0$, $I^{(m)}$ contains more than 2 elements. So T_2 contains no terms of positive degree. Hence T_2 preserves $\langle 1, x, x^2, \dots, x^n \rangle$ for all n. This type of operators is already studied in [4] called there exactly-solvable operators), and we do not repeat it here.
(2) $I^{(m)}$ is empty. This can only happen, if $m = 0$. But this case is trivial, since all operators of degree 0 are in $\mathfrak{D}_V$. Hence degree 0 contributes to T_2 the operator $\alpha_1 x^2 \partial^2 + \alpha_2 x \partial + \alpha_3$. This part we call trivial, and is always present in T_2.
(3) $I^{(m)}$ contains one element. Let us assume here that $i_1 > i_2 > \dots > i_n$. Then we have $i_j = i_{j-1} + m$ so that $i_j = (n-j)m$ (since we assumed that $1 \in V$, so $i_n = 0$). But we also assumed that the i_j have no common factor, so it follows that $m = 1$, and hence $V = \langle 1, x, x^2, \dots, x^{n-1} \rangle$. Hence we are in the case that is extensively discussed in [4].

(4) $I^{(m)}$ contains 2 elements, $m > 0$, and no $I^{(l)}$ for $l > 0$ contains one element. Suppose i_1 and i_2 are the 2 elements in $I^{(m)}$. Then the set $\{i_1, i_2, \dots, i_n\}$ is of the following specific form:

$$i_3 = i_1 - m,\ i_5 = i_3 - m,\ \dots,\ i_{2r-1} = i_{2r-3} - m$$

and

$$i_4 = i_2 - m,\ i_6 = i_4 - m,\ \dots,\ i_{2s} = i_{2s-2} - m$$

We call $i_1, i_3, \dots, i_{2r-1}$ and $i_2, i_4, \dots, i_{2s}$ *chains* with step m and length r and s, respectively. In general, with no special relation, by which we mean that I can be split into two chains in exactly one way, the most general second-order operator in $\mathfrak{D}_V$ is

$$T_2 = \alpha_1 x^m (x\partial - i_1)(x\partial - i_2) + \alpha_2 x^2 \partial^2 + \alpha_3 x\partial + \alpha_4$$

and T_2 contains an extra term in two cases:

(a) If $m = 1$, (and therefore $i_{2s} = 0$), we get an extra term $\alpha_5 \partial(x\partial - i_{2r-1})$.

(b) If m=2 and $\{i_{2s}, i_{2r-1}\} = \{0, 1\}$, we get an extra term $\alpha_5 \partial^2$.

All this can easily be proved, by examining the possibilities for which $I^{(m)}$ can have 0,1 or 2 elements.

5.2. Special subspaces. For existence of non-trivial second order operators in $\mathfrak{D}_V$, it is necessary that I can be split into 2 chains. The form of T_2 above is under the assumption that I can be split into two chains in exactly one way. There are cases, where I can be split in more than one way. As an example consider $I = \{0, 1, 2, \dots, 98, 100\}$. Then $I = \{0, 1, 2, \dots, 98\} \cup \{100\}$ or $I = \{0, 2, 4, \dots, 98, 100\} \cup \{1, 3, 5, \dots, 97\}$. Consequently the space V admits a more general second-order operator. Here we describe all special cases, which fall into 4 groups:

Case A. The dimension of V is 3, so $I = \{0, m, m + l\}$, $l \neq m$. Then

$$\begin{aligned} T_2 &= \alpha_1 x^{l+m} (x\partial - m)(x\partial - l - m) + \alpha_2 x^{l+1} \partial (x\partial - l - m) \\ &\quad + \alpha_3 x^m (x\partial - m)(x\partial - l - m) + \alpha_4 x^2 \partial^2 + \alpha_5 x\partial + \alpha_6 \end{aligned}$$

T_2 gets a certain extra terms in the following cases:

(a) $m = 1$, $l > 2$. Extra term $\alpha_7 \partial(x\partial - l - m)$.

(b) $m = 1$, $l = 2$. Extra terms $\alpha_7 \partial(x\partial - 3) + \alpha_8 \partial^2$.

(c) $l = 1$ (hence $m > 1$). Extra term $\alpha_7 \partial(x\partial - m)$.

Case B. The dimension of V is 4, and I is "symmetric", i.e. $I = \{0, m, m+l, 2m+l\}$, $l \neq m$. Then

$$\begin{aligned} T_2 &= \alpha_1 x^{l+m} (x\partial - 2m - l)(x\partial - l - m) + \alpha_2 x^m (x\partial - 2m - l)(x\partial - m) \\ &\quad + \alpha_3 x^2 \partial^2 + \alpha_4 x\partial + \alpha_5 \end{aligned}$$

T_2 gets an extra term only if

(a) $m = 1$. Extra term $\alpha_6 \partial(x\partial - m - l)$.

Case C. I has one runner ahead at distance 2, i.e. $I = \{0, 1, 2, \dots, n-2, n\}$. Then T_2 takes the form

$$\begin{aligned} T_2 = \alpha_1 x^2(x\partial - n)(x\partial - n + 3) + \alpha_2 x(x\partial - n)(x\partial - n + 2) \\ + \alpha_3 x^2\partial^2 + \alpha_4 x\partial + \alpha_5 \\ + \alpha_6\partial(x\partial - n) + \alpha_7\partial^2 \end{aligned}$$

Case D. I has one runner left behind at distance 2, i.e. $I = \{0, 2, 3, 4, \dots, n\}$. Then T_2 is of the form

$$\begin{aligned} T_2 = \alpha_1 x^2(x\partial - n)(x\partial - n + 1) + \alpha_2 x^2\partial(x\partial - n) \\ + \alpha_3 x^2\partial^2 + \alpha_4 x\partial + \alpha_5 \\ + \alpha_6\partial(x\partial - 2) \end{aligned}$$

These special cases exhaust the list of all exceptional cases, which do not belong to the general classification given above. Let us prove this. It is clear that we can assume that the dimension is 5 or more, since the dimensions 3 and 4 are considered above. We know that I can be split into two chains of step, say, m, so $i_3 = i_1 - m$, $i_5 = i_3 - m$, $\dots$, i_{2r-1} and $i_4 = i_2 - m$, $i_6 = i_4 - m$, $\dots$, i_{2s}. Suppose that I can also be split in two chains of step l. We may assume that $l > m$. Now we distinguish three cases:

(1) $r = 1$, so i_3 is not present.
We have either $I^{(l)} \supset \{i_1, i_2\}$ or $I^{(l)} \supset \{i_1, i_4\}$. Therefore we consider two subcases:
   a. $I^{(l)} \supset \{i_1, i_2\}$, so $l \neq i_1 - i_2$. Since $\mid I^{(l)} \mid= 2$, we have $i_4 \notin I^{(l)}$, so $i_1 - i_4 = l$. But then $i_1 - i_6 \neq l$, and therefore $i_2 - i_6 = 2m = l$. This is not allowed, since $i_1 - i_4 = 2m$ implies that $i_1 - i_2 = m$.
   b. $I^{(l)} \supset \{i_1, i_4\}$, so $i_1 - i_2 = l$. But also $i_6 \notin I^{(l)}$, and this is only possible if $i_2 - i_6 = l = 2m$. Therefore this configuration leads to case C.

(2) $s = 1$, so i_4 is not present.
Clearly $I^{(l)} \supset \{i_1, i_3\}$. Moreover $i_2 < i_3$ since otherwise also $i_2 \in I^{(l)}$. But then $i_5 \notin I^{(l)}$ implies that $i_5 + l = i_1$, and hence $l = 2m$. Again using $i_2 \notin I^{(l)}$ implies that $i_2 + l = i_{2r-1}$. Therefore this configuration leads to case D.

(3) $r > 1$ and $s > 1$, so i_3 and i_4 are both present.
Always $I^{(l)} \supset \{i_1, i_3\}$, and necessarily $i_2 < i_3$, since otherwise also $i_2 \in I^{(l)}$. Again we have two subcases:
   a. i_5 is present. We need $i_5 + l = i_1$, so $l = 2m$. Like in case 2 above, it follows that $i_2 = i_{2r-1} - 2m$. But then $i_4 \in I^{(l)}$.
   b. i_6 is present, but not i_5. We need either $i_1 - i_2 = l$ or $i_3 - i_2 = l$. If $i_1 - i_2 = l$, then $i_6 + l = i_2$ is the only possibility, so $l = 2m$, and hence $i_3 - i_2 = m$. This is forbidden because then we have one chain. If $i_3 - i_2 = l$, then $i_4 + l \notin \{i_1, i_3, i_2\}$, so $i_4 \in I^{(l)}$. So this is also impossible.

6. Conclusion

In the previous sections we considered the case that the powers of x are natural numbers. From algebraic point of view, this is not a crucial restriction. One could take the powers to be all integer, rational, real or complex numbers, or any other abelian subgroup of $\mathbb{C}$. (If one takes $\mathbb{C}$, one has to define a suitable total ordering on $\mathbb{C}$ to be able to consider positive and negative). This has as a main advantage that such algebras of "polynomials" admit the isomorphism $x \mapsto 1/x$. Moreover, the map $x^k \mapsto x^{k+l}$ is a linear isomorphism; it is the gauge transformation discussed before.

An even more special case is that the set of allowed powers form a field. In this case, the change of basis $x \mapsto x^m$ is an isomorphism.

We shortly discuss the case that the powers of x are real numbers. The algebra of "generalized polynomials", which we denote (suggestively) by $\mathbb{C}[x^\alpha]$ is as a linear space

$$\bigoplus_{\alpha\in\mathbb{R}} \mathbb{C}\cdot x^\alpha$$

and the multiplication is given by

$$x^\beta \cdot x^\gamma = x^{\beta+\gamma} \qquad \beta,\gamma\in\mathbb{R}$$

Let $\mathfrak{D}$ denote the algebra of differential operators with coefficients in $\mathbb{C}[x^\alpha]$ now. The degree of $T\in\mathfrak{D}$ can be any real number. For $T\in\mathfrak{D}$, $\deg(T)=m$ we have

$$T=\sum_{i=0}^{k} c_i x^{i+m}\partial^i.$$

with $c_i\in\mathbb{C}$, and $c_k\neq 0$. Lemma 3.1 for this operator T now looks like:

Lemma 6.1. *There exist numbers $\alpha_1,\dots,\alpha_k\in\mathbb{C}$ such that*

$$T=c_k x^m(x\partial-\alpha_1)(x\partial-\alpha_2)\dots(x\partial-\alpha_k).$$

In particular, the case that $m<0$ dissappears, or, better, ∂^{-m} factorizes:

$$\partial^s = x^{-s}(x\partial-(s-1))(x\partial-(s-2))\cdots(x\partial-1)x\partial \qquad (s>0)$$

We denote $I=\{i_1,i_2,\dots,i_n\}$, where $i_j\in\mathbb{R},\ j=1,\dots,n$, and define for $m\in\mathbb{R}$

$$I^{(m)}=\{i\in I \mid i+m\notin I\}.$$

Let $V=\langle x^{i_1},x^{i_2},\dots,x^{i_n}\rangle$ and let $\mathfrak{D}_V$ denote the algebra of operators in $\mathfrak{D}$ that leave V invariant. One can easily check that Theorem 3.2 holds literally. Due to the remarks at the beginning of this section, we have the following.

Proposition 6.2.

1. *The gauge transformation $T\mapsto x^l T x^{-l}$ gives an isomorphism between $\mathfrak{D}_V$ and $\mathfrak{D}_W$, with $V=\langle x^{i_1},x^{i_1},\dots,x^{i_n}\rangle$ and $W=\langle x^{i_1-l},x^{i_2-l},\dots,x^{i_n-l}\rangle$.*
2. *The change of basis $x'=x^m$ induces an isomorphism between $\mathfrak{D}_V$ and $\mathfrak{D}_W$, where $V=\langle x^{i_1},x^{i_1},\dots,x^{i_n}\rangle$ and $W=\langle x^{i_1/m},x^{i_2/m},\dots,x^{i_n/m}\rangle$.*

Proof.

(1) Part (1) is obvious.

(2) By the change of basis $x' = x^m$ we have that

$$x\partial = m\, x'\partial', \qquad \partial' = \frac{d}{dx'}.$$

Hence the operator $T = \tilde{T}\cdot(x\partial-\alpha_1)(x\partial-\alpha_2)\cdots(x\partial-\alpha_k)$, where $\deg(\tilde{T}) = l$ is mapped to $T' = \tilde{T'}\cdot(mx'\partial'-\alpha_1)(mx'\partial'-\alpha_2)\cdots(mx'\partial'-\alpha_k)$ with $\deg(\tilde{T'}) = l/m$. This leads directly to the proof of statement (2) of the Proposition. □

The classification of second order operators $T_2 \in \mathfrak{D}_V$ is similar as before. A difference is that $|I^{(m)}| = |I^{(-m)}|$, so that there is always a sort of symmetry in T_2. Here we give one example, the generic 2-chain case. Suppose $|I^{(m)}| = 2$, so I has the structure $I = \{i_1, i_3, \dots, i_{2r-1}\} \cup \{i_2, i_4, \dots, i_{2s}\}$ with

$$i_3 = i_1 - m,\ i_5 = i_3 - m,\ \dots,\ i_{2r-1} = i_{2r-3} - m$$

and

$$i_4 = i_2 - m,\ i_6 = i_4 - m,\ \dots,\ i_{2s} = i_{2s-2} - m$$

If for all $l > 0, l \neq m$ there holds $|I^{(l)}| > 2$, then the most general 2^{nd} order operator $T_2 \in \mathfrak{D}_V$ takes the form:

$$T_2 = \alpha_1 x^m(x\partial - i_1)(x\partial - i_2) + \alpha_2 x^2\partial^2 + \alpha_3 x\partial + \alpha_4 + \alpha_5 x^{-m}(x\partial - i_{2r-1})(x\partial - i_{2s})$$

Note the symmetry in T_2: there are as many terms of degree m as terms of degree $-m$.

Final remark. In the case that $V = \langle 1, x, x^2, \dots, x^n\rangle$, the algebra $\mathfrak{D}_V$ is essentially generated by $\{\partial, 2x\partial - n, x^2\partial - nx\}$, see [4]. These 3 elements form the Lie algebra $sl_2(\mathbb{C})$, and hence $\mathfrak{D}_V$ is essentially some representation of the universal enveloping algebra of $sl_2(\mathbb{C})$. We were not able to find a similar structure in $\mathfrak{D}_V$ for general V, even for the simplest case $V = \langle 1, x, x^3\rangle$. Particularly, one can show that for the space $V = \langle 1, x, x^3\rangle$ the algebra $\mathfrak{D}_V$ is an infinite-dimensional, finite-generated algebra. It is defined by 11 generators, which are differential operators of first-, second- and third order, and the commutator of any two of them is expressible as an *ordered* cubic polynomial in these generators.

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